A BGK approximation to scalar conservation laws with discontinuous flux

F. Berthelin*and J. Vovelle†

Abstract

We study the BGK approximation to first-order scalar conservation laws with a flux which is discontinuous in the space variable. We show that the Cauchy Problem for the BGK approximation is well-posed and that, as the relaxation parameter tends to 0, it converges to the (entropy) solution of the limit problem.

Keywords: scalar conservation laws – discontinuous flux – BGK model – relaxation limit

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1 Introduction

In this paper we consider the equation

$$\partial_t f^\varepsilon + \partial_x(k(x)a(\xi)f^\varepsilon) = \frac{\chi_{u^\varepsilon} - f^\varepsilon}{\varepsilon}, \quad t > 0,\, x \in \mathbb{R},\, \xi \in \mathbb{R}, \tag{1}$$

with the initial condition

$$f^\varepsilon|_{t=0} = f_0, \text{ in } \mathbb{R}_x \times \mathbb{R}_\xi. \tag{2}$$

Here k is given by

$$k = k_L \mathbf{1}_{(-\infty,0)} + k_R \mathbf{1}_{(0,+\infty)},$$

where $\mathbf{1}_B$ is the characteristic function of a set B, $\xi \mapsto a(\xi)$ is a continuous function on $\mathbb{R}$ such that

$$\forall u \in [0,1], \int_0^u a(\xi)d\xi \geq 0, \quad \int_0^1 a(\xi)d\xi = 0, \tag{3}$$

and, in (1), χ_{u^ε}, the so-called *equilibrium function* associated to f^ε is defined by

$$u^\varepsilon(t,x) = \int_{\mathbb{R}} f^\varepsilon(t,x,\xi)d\xi, \quad \chi_\alpha(\xi) = \mathbf{1}_{]0,\alpha[}(\xi) - \mathbf{1}_{]\alpha,0[}(\xi),$$

*Laboratoire J. A. Dieudonné, UMR 6621 CNRS, Université de Nice, Parc Valrose, 06108 Nice cedex 2, France, Email: Florent.Berthelin@unice.fr

†Université de Lyon ; CNRS ; Université Lyon 1, Institut Camille Jordan, 43 boulevard du 11 novembre 1918, F-69622 Villeurbanne Cedex, France. Email: vovelle@math.univ-lyon1.fr

for $t > 0, x \in \mathbb{R}, \xi \in \mathbb{R}, \alpha \in \mathbb{R}$.

Eq. (1) is the so-called BGK approximation to the scalar conservation law

$$\partial_t u + \partial_x(k(x)A(u)) = 0, \quad A(u) = \int_0^u a(\xi)d\xi. \tag{4}$$

The flux $(x, u) \mapsto k(x)A(u)$ is discontinuous with respect to $x \in \mathbb{R}$, actually (4) is a prototype of scalar (first-order) conservation law with discontinuous flux function. In the last ten years, scalar conservation laws with discontinuous flux function have been extensively studied. We refer to the paper [BK08] for a comprehensive introduction to the subject and a complete list of references. Let us simply mention that the discontinuous character of the flux function gives rise to a multiplicity of weak solutions, even if traditional entropy conditions are imposed in the spatial domain apart from the discontinuity. An additional criterion has therefore to be given in order to select solutions in a unique way. For scalar conservation law under the general form $\partial_t u + \partial_x(B(x, u)) = 0$, where the function B is discontinuous with respect to x, several criteria are possible [AMG05]. For $B(x, u) = k(x)A(u)$ as above, the choice of entropy solution is unambiguous (see [AMG05], Remark 4.4) and we consider here the criterion of selection first given in [Tow01]. A kinetic formulation (in the spirit of [LPT94]) equivalent to the entropy formulation in [Tow01] has been given in [BV06]. In particular, solutions given by this criterion are limits (a.e. and in L^1) of the solutions obtained by *monotone* regularization of the coefficient k in (4), *e.g.*

$$k_\varepsilon(x) = k_L \mathbf{1}_{x<-\varepsilon}(x) + \left(\frac{k_R - k_L}{2\varepsilon} x + \frac{k_R + k_L}{2} \right) \mathbf{1}_{-\varepsilon \le x \le \varepsilon} + k_R \mathbf{1}_{\varepsilon < x}, \quad \varepsilon > 0.$$

The kinetic formulation of scalar conservation laws is well adapted to the analysis of the (Perthame-Tadmor) BGK approximation of scalar conservation laws. Developed in [PT91], this equation is a continuous version of the Transport-Collapse method of Brenier [Bre81, Bre83]. BGK models have also been used for gas dynamics and the construction of numerical schemes. See for example the book of Perthame [Per02] for a survey of this field.

Our purpose here is to apply the kinetic formulation of [BV06] to show the convergence of the BGK approximation. To this aim, we first study the BGK equation in itself in Section 2. In Section 3, we introduce the kinetic formulation for the limit problem. We also introduce a notion of generalized (kinetic) solution, Definition 6. We show that any generalized solution reduces to a mere solution, *i.e.* a solution in the sense of Def. 4. This theorem of "reduction" is Theorem 7. Then in Section 4, we show that the BGK model converges to a generalized solution of (4) and, using Theorem 7, deduce the strong convergence of the BGK model to a solution of (4), Theorem 11.

A key step of the whole proof of convergence is the result of reduction of Theorem 7. Its proof, given in Section 3.2, is close to the proof of uniqueness of solutions given in [BV06]. A minor difference is that we deal here with generalized solutions instead of "kinetic process solutions". There is also a minor error in the proof given in [BV06] (specifically, the remainder terms $R_{\alpha,\varepsilon,\delta}$ and $Q_{\beta,\nu,\sigma}$ in Eq. (31) and (32) of the present paper are missing in [BV06]). We have therefore given a complete proof of Theorem 7.

We end this introduction with two remarks:

- the BGK model provides an approximation of the entropy solutions to (4) by relaxation of the kinetic equation corresponding to (4). A relaxation scheme of the Jin and Xin type applied directly to the original equation (4) has been developed in [KKR04].
- in the last chapter of [Bac05], is derived the kinetic formulation of scalar conservation laws with discontinuous spatial dependence of the form $\partial_t u + \partial_x(B(x,u)) = 0$ (which are more general than (4)). We indicate (this would have to be proved rigorously), that in case where our approach *via* the BGK approximation was applied to this problem, the solutions obtained would be the type of entropy solutions considered in [KRT03].

Notation For $p, q \in [1, +\infty]$, we denote by $L^p_x L^q_\xi$ the space $L^p(\mathbb{R}_x; L^q(\mathbb{R}_\xi))$ and by $L^q_\xi L^p_x$ the space $L^q(\mathbb{R}_\xi; L^p(\mathbb{R}_x))$.
We also set $\mathrm{sgn}_+(s) = \mathbf{1}_{\{s>0\}}$, $\mathrm{sgn}_-(s) = -\mathbf{1}_{\{s\leq 0\}}$, $\mathrm{sgn} = \mathrm{sgn}_+ + \mathrm{sgn}_-$, $s \in \mathbb{R}$.

2 The BGK equation

2.1 The balance equation

By the change of variables $\tilde{f}^\varepsilon(t,x,\xi) = e^{\frac{t}{\varepsilon}} f^\varepsilon(t,x,\xi)$, Eq. (1) rewrites as the balance equation

$$\partial_t \tilde{f}^\varepsilon + \partial_x(k(x)a(\xi)\tilde{f}^\varepsilon) = \frac{e^{\frac{t}{\varepsilon}}}{\varepsilon}\chi_{u^\varepsilon}$$

with (unknown dependent) source term $\frac{e^{\frac{t}{\varepsilon}}}{\varepsilon}\chi_{u^\varepsilon}$. Hence, we first consider the following Cauchy Problem for the balance equation:

$$\partial_t f + \partial_x(k(x)a(\xi)f) = g, \quad t > 0, x \in \mathbb{R}, \xi \in \mathbb{R}, \tag{5}$$

$$f|_{t=0} = f_0 \quad \text{in } \mathbb{R}_x \times \mathbb{R}_\xi. \tag{6}$$

Proposition 1 *Suppose that $k_R \cdot k_L > 0$. Then Problem (5)-(6) is well posed in $L^1_\xi L^p_x$, $1 \leq p < +\infty$: for all $f_0 \in L^1_\xi L^p_x$, $T > 0$ and $g \in L^1(]0,T[; L^1_\xi L^p_x)$, there exists a unique $f \in C([0,T]; L^1_\xi L^p_x)$ solving (5) in $\mathcal{D}'(]0,T[\times\mathbb{R}_x \times \mathbb{R}_\xi)$ such that $f(0) = f_0$. Besides, we have*

$$\|f(t)\|_{L^1_\xi L^p_x} \leq M_k \left(\|f_0\|_{L^1_\xi L^p_x} + \int_0^t \|g(s)\|_{L^1_\xi L^p_x} ds \right), \tag{7}$$

where $M_k = \max\left(\frac{k_L}{k_R}, \frac{k_R}{k_L}\right)$.

Proof: Since (5) is linear, it is sufficient to solve the case $g = 0$. The general case will follow from Duhamel's Formula. Assume without loss of generality $k_R, k_L > 0$. Let $A_+ := \{\xi \in \mathbb{R}; a(\xi) > 0\}$. Then, for fixed $\xi \in A_+$, and although k is a discontinuous function, the O.D.E.

$$\dot{X}(t,s,x,\xi) = k(X(t,s,x,\xi))a(\xi), \quad t \in \mathbb{R}, \tag{8}$$

with datum $X(s,s,x,\xi)=x$ has an obvious solution for $x\neq 0$, given by $X(t,s,x,\xi)=x+(t-s)k_Ra(\xi)$, $t>s$, when $x>0$, and by

$$X(t,s,x,\xi)=\begin{cases} x+(t-s)k_La(\xi) & \text{if} \quad t<s+\frac{|x|}{k_La(\xi)}, \\ \frac{k_R}{k_L}x+(t-s)k_Ra(\xi) & \text{if} \quad t>s+\frac{|x|}{k_La(\xi)}, \end{cases}$$

when $x<0$. Denoting by $s^+=\max(s,0)$, $s^-=s^+-s$ the positive and negative parts of $s\in\mathbb{R}$, and introducing

$$\alpha_k(x)=\mathbb{1}_{\{x>0\}}+\frac{k_R}{k_L}\mathbb{1}_{\{x<0\}},$$

this can be summed up as

$$X(t,s,x,\xi)=\{\alpha_k(x)x+(t-s)k_Ra(\xi)\}^+-\{x+(t-s)k_La(\xi)\}^-, \quad t>s. \tag{9}$$

Similarly, we have, for the resolution of (8) backward in time,

$$X(t,s,x,\xi)=\{x+(t-s)k_Ra(\xi)\}^+-\{\beta_k(x)x+(t-s)k_La(\xi)\}^-, \quad t<s, \tag{10}$$

where

$$\beta_k(x)=\frac{k_L}{k_R}\mathbb{1}_{\{x>0\}}+\mathbb{1}_{\{x<0\}}.$$

A similar computation in the case $a(\xi)\leq 0$ gives the solution to (8) by (9) for $(t-s)a(\xi)\geq 0$, (10) for $(t-s)a(\xi)\leq 0$. For the transport equation $(\partial_t+k(x)a(\xi)\partial_x)\varphi^*=0$, interpreted as

$$\frac{d}{dt}\varphi^*(t,X(t,s,x,\xi),\xi)=0,$$

this yields the solution

$$\varphi^*(t,x,\xi)=\psi(X(T,t,x,\xi),\xi),$$

which satisfies the terminal condition $\varphi^*(T)=\psi$. We suppose in what follows that ψ is independent on ξ, compactly supported and Lipschitz continuous. Then, a simple change of variable shows that, for every $t\in[0,T]$, for a.e. $\xi\in\mathbb{R}$,

$$\|\varphi^*(t,\cdot,\xi)\|_{L^q_x}\leq M_k\|\psi\|_{L^q_x}, \quad M_k=\max\left(\frac{k_L}{k_R},\frac{k_R}{k_L}\right), 1\leq q\leq+\infty. \tag{11}$$

If $f\in C([0,T];L^1_\xi L^p_x)$ solves (5)-(6), then, by duality (note that φ^* is Lipschitz continuous and compactly supported in x if ψ is) we have, for $t\in[0,T]$, for a.e. $\xi\in\mathbb{R}$,

$$\int_{\mathbb{R}} f(T,x,\xi)\psi(x,\xi)dx=\int_{\mathbb{R}} f_0(x,\xi)\varphi^*(0,x,\xi)dx. \tag{12}$$

In particular, the estimate (11) where q = conjugate exponent of p gives, for a.e. $\xi\in\mathbb{R}$,

$$\|f(T,\cdot,\xi)\|_{L^p_x}\leq M_k\|f_0(\cdot,\xi)\|_{L^p_x},$$

and then by Duhamel's principle, for $g\neq 0$,

$$\|f(T,\cdot,\xi)\|_{L^p_x}\leq M_k\left(\|f_0(\cdot,\xi)\|_{L^p_x}+\int_0^T\|g(t,\cdot,\xi)\|_{L^p_x}dt\right). \tag{13}$$

The estimate (7) and uniqueness of the solution to (5)-(6) readily follows. Existence follows from (9)-(10)-(12), from which one derives the explicit formula

$$f(t,x,\xi)=J(t,x,\xi)f_0(X(0,t,x,\xi),\xi),$$

the coefficient $J(t,x,\xi)$ being given by

$$J(t,x,\xi)=\mathbb{1}_{\{x<0\}\cup\{x>tk_Ra(\xi)\}}+\frac{k_L}{k_R}\mathbb{1}_{\{0<x<tk_Ra(\xi)\}}$$

if $a(\xi)>0$ and

$$J(t,x,\xi)=\mathbb{1}_{\{x<tk_La(\xi)\}\cup\{x>0\}}+\frac{k_R}{k_L}\mathbb{1}_{\{tk_La(\xi)<x<0\}}$$

if $a(\xi)\leq 0$.

2.2 The BGK equation

Denote by $\mathcal{T}(t)f_0$ the solution to (5)-(6) with $g=0$, *i.e.*

$$\mathcal{T}(t)f_0(x,\xi)=J(t,x,\xi)f_0(X(0,t,x,\xi),\xi),$$

X given by (9)-(10).

Definition 2 *Let $f_0\in L^1(\mathbb{R}_x\times\mathbb{R}_\xi)$, $T>0$. A function $f^\varepsilon\in C([0,T];L^1(\mathbb{R}_x\times\mathbb{R}_\xi))$ is said to be a solution to (1)-(2) if*

$$f^\varepsilon(t)=e^{-\frac{t}{\varepsilon}}\mathcal{T}(t)f_0+\frac{1}{\varepsilon}\int_0^t e^{-\frac{s}{\varepsilon}}\mathcal{T}(s)\chi_{u^\varepsilon(t-s)}ds,\quad u^\varepsilon=\int_{\mathbb{R}}f^\varepsilon(\xi)d\xi,\tag{14}$$

for all $t\in[0,T]$.

Theorem 3 *Assume $k_R\cdot k_L>0$. Let $f_0\in L^1(\mathbb{R}_x\times\mathbb{R}_\xi)$, $T>0$. There exists a unique solution $f^\varepsilon\in C([0,T];L^1(\mathbb{R}_x\times\mathbb{R}_\xi))$ to (1)-(2). Denoting by $S_\varepsilon(t)f_0$ this solution, we have:*

1. $\|(S_\varepsilon(t)f_0^\natural-S_\varepsilon(t)f_0^\flat)^+\|_{L^1(\mathbb{R}_x\times\mathbb{R}_\xi)}\leq M_k\|(f_0^\natural-f_0^\flat)^+\|_{L^1(\mathbb{R}_x\times\mathbb{R}_\xi)}$
2. $0\leq \operatorname{sgn}(\xi)f_0(x,\xi)\leq 1$ *a.e.* $\Rightarrow 0\leq\operatorname{sgn}(\xi)S_\varepsilon(t)f_0(x,\xi)\leq 1$ *a.e.*
3. *if* $f_0=\chi_{u_0}$, $u_0\in L^\infty(\mathbb{R})$, $0\leq u_0\leq 1$ *a.e. then* $0\leq S_\varepsilon(t)f_0\leq\chi_1$.

Proof: the change of variable $(t',x')=\varepsilon(t,x)$ reduces (1) to the same equation with $\varepsilon=1$. We then have to solve $f=F(f)$ for

$$F(f)(t):=e^{-t}\mathcal{T}(t)f_0+\int_0^t e^{-s}\mathcal{T}(s)\chi_{u(t-s)}ds,\quad u=\int_{\mathbb{R}}f(\xi)d\xi.$$

By (7) and the identity

$$\int_{\mathbb{R}}|\chi_u-\chi_v|(\xi)d\xi=|u-v|,\quad u,v\in\mathbb{R},$$

we have $F\colon C([0,T];L^1_{x,\xi}) \to C([0,T];L^1_{x,\xi})$ and F is a $(1-e^{-T})$ contraction for the norm

$$\|f\| = \sup_{t\in[0,T]} \|f(t)\|_{L^1(\mathbb{R}_x\times\mathbb{R}_\xi)}.$$

Indeed, we compute,

$$\begin{aligned}
\|F(f^\natural)(t)-F(f^\flat)(t)\|_{L^1_{x,\xi}} &\leq \int_0^t e^{-s}\|\mathcal{T}(s)(\chi_{u^\natural(t-s)}-\chi_{u^\flat(t-s)})\|_{L^1_{x,\xi}} ds\\
&= \int_0^t e^{-s}\|\chi_{u^\natural(t-s)}-\chi_{u^\flat(t-s)}\|_{L^1_{x,\xi}} ds\\
&= \int_0^t e^{-s}\|u^\natural(t-s)-u^\flat(t-s)\|_{L^1_x} ds\\
&\leq \int_0^t e^{-s}\|f^\natural(t-s)-f^\flat(t-s)\|_{L^1_{x,\xi}} ds\\
&\leq \int_0^t e^{-s}ds\|f^\natural-f^\flat\|.
\end{aligned}$$

By the Banach fixed point theorem, we obtain existence and uniqueness of the solution to (1)-(2). Since $0\leq \operatorname{sgn}(\xi)\chi_u(\xi)\leq 1$ a.e. we have

$$0\leq \operatorname{sgn}(\xi)F(f)(t,x,\xi)\leq 1 \text{ a.e.}$$

if $0\leq \operatorname{sgn}(\xi)f_0(x,\xi)\leq 1$ a.e. This proves the point *2.* of the Theorem. The point *1.* follows from the inequality

$$\int_{\mathbb{R}} \operatorname{sgn}_+(f-g)(Q(f)-Q(g))d\xi \leq 0, \quad f,g\in L^1(\mathbb{R}_\xi), \quad Q(f):=\chi_{\int fd\xi}-f,$$

that is easy to check, and from the identity

$$f(t)=\mathcal{T}(t)f_0+\int_0^t \mathcal{T}(s)Q(f)(t-s)ds$$

for the solution to (1)-(2). If $f_0=\chi_{u_0}$, $0\leq u_0\leq 1$ a.e. then $0=\chi_0\leq f_0\leq\chi_1$. Hence the item *3.* follows from *1.* and the fact that any constant equilibrium function χ_α, $\alpha\in\mathbb{R}$ is solution to (1). ■

3 The limit problem

Assume $f_0=\chi_{u_0}$ with $u_0\in L^\infty(\mathbb{R})$, $0\leq u_0\leq 1$ a.e. Set

$$A(u)=\int_0^u a(\xi)\mathbf{1}_{[0,1]}(\xi)d\xi. \tag{15}$$

Note that by (3), we have $A\geq 0$ and A vanishes outside the interval $[0,1]$. We expect the solution f^ε to (1)-(2) to converge to the solution u of the first-order scalar conservation law

$$\partial_t u+\partial_x(k(x)A(u))=0, \quad t>0, x\in\mathbb{R}, \tag{16}$$

with initial datum

$$u(0,x)=u_0(x), \quad x\in\mathbb{R}. \tag{17}$$

For a fixed $T>0$, set $Q=]0,T[\times\mathbb{R}_x$.

Definition 4 (Solution) *Let $u_0 \in L^\infty(\mathbb{R})$, $0 \le u_0 \le 1$ a.e. A function $u \in L^\infty(Q)$ is said to be a (kinetic) solution to (16)-(17) if there exists non-negative measures $m_\pm$ on $[0,T] \times \mathbb{R} \times \mathbb{R}$ such that*

- *m_+ is supported in $[0,T] \times \mathbb{R} \times]-\infty, 1]$, m_- is supported in $[0,T] \times \mathbb{R} \times [0, +\infty[$,*
- *for all $\psi \in C_c^\infty([0,T[\times \mathbb{R} \times \mathbb{R})$,*

$$\int_Q \int_{\mathbb{R}} h_\pm (\partial_t \psi + k(x) a(\xi) \partial_x \psi) d\xi dt dx$$
$$+ \int_{\mathbb{R}} \int_{\mathbb{R}} h_{0,\pm} \psi(0,x,\xi) d\xi dx - (k_L - k_R)^\pm \int_0^T \int_{\mathbb{R}} a(\xi) \psi(t,0,\xi) d\xi dt$$
$$= \int_Q \int_{\mathbb{R}} \partial_\xi \psi dm_\pm(t,x,\xi) \quad (18)$$

where $h_\pm(t,x,\xi) = \operatorname{sgn}_\pm(u(t,x) - \xi)$, $h_{0,\pm}(x,\xi) = \operatorname{sgn}_\pm(u_0(x) - \xi)$.

Proposition 5 (Bound in L^∞) *Let $u_0 \in L^\infty(\mathbb{R})$, $0 \le u_0 \le 1$ a.e. If $u \in L^\infty(Q)$ is a kinetic solution to (16)-(17), then $0 \le u \le 1$ a.e.*

Proof: Consider the kinetic formulation (18) for h_+ with a test function

$$\psi(t,x,\xi) = \varphi(t,x)\mu(\xi).$$

If μ is supported in $]1, +\infty[$, two terms cancel:

$$\int_{\mathbb{R}} \int_{\mathbb{R}} h_{0,+} \psi(0,x,\xi) d\xi dx = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbf{1}_{1 \ge u_0(x) > \xi} \varphi(0,x) \mathbf{1}_{\xi > 1} \mu(\xi) d\xi dx = 0$$

and

$$\int_Q \int_{\mathbb{R}} \partial_\xi \psi dm_+(t,x,\xi) = 0$$

by the hypothesis on the support of m_+. Hence we have

$$\int_Q \int_{\mathbb{R}} h_+ (\partial_t \varphi + k(x) a(\xi) \partial_x \varphi) \mu(\xi) d\xi dt dx$$
$$- (k_L - k_R)^+ \int_0^T \int_{\mathbb{R}} a(\xi) \varphi(t,0) \mu(\xi) d\xi dt = 0.$$

A step of approximation and regularization shows that we can take $\mu(\xi) = \mathbf{1}_{\xi > 1}$ in this equation. Since

$$\int_1^{+\infty} a(\xi) d\xi = A(+\infty) - A(1) = 0 - 0 = 0,$$

and

$$\int_1^{+\infty} h_+(t,x,\xi) d\xi = \int_1^{+\infty} \mathbf{1}_{\xi < u(t,x)} d\xi = (u(t,x) - 1)^+,$$

$$\int_1^{+\infty} h_+(t,x,\xi)a(\xi)d\xi = \int_1^{+\infty} \mathbf{1}_{\xi<u(t,x)}a(\xi)d\xi$$

$$= \mathrm{sgn}_+(u(t,x)-1)\int_1^{u(t,x)} a(\xi)d\xi = \mathrm{sgn}_+(u(t,x)-1)(A(u(t,x)-A(1)),$$

we obtain

$$\int_Q (u-1)^+\partial_t\varphi + k(x)\mathrm{sgn}_+(u-1)(A(u)-A(1))\partial_x\varphi dtdx = 0.$$

It is then classical to deduce that $(u-1)^+ = 0$ a.e. (see the end of the proof of Proposition 10, after (39)), *i.e.* $u \leq 1$ a.e. Similarly, we show $u \geq 0$ a.e. ■

Our aim is to prove the uniqueness of the solution to (16)-(17). Actually, more than mere uniqueness of the solution to (16)-(17), we will show a result of reduction/uniqueness (see Theorem 7) of generalized kinetic solution. To this purpose, let us recall that a Young measure $Q \to \mathbb{R}$ is a measurable mapping $(t,x) \mapsto \nu_{t,x}$ from Q into the space of probability (Borel) measures on $\mathbb{R}$. The mapping is measurable in the sense that for each Borel subset A of $\mathbb{R}$, $(t,x) \mapsto \nu_{t,x}(A)$ is measurable $Q \to \mathbb{R}$. Let us also introduce the following notation: if $f \in L^1(Q\times\mathbb{R})$, we set

$$f_\pm(y,\xi) = f(y,\xi) - \mathrm{sgn}_\mp(\xi), \quad y \in Q, \xi \in \mathbb{R}.$$

This is consistent with the notations used in Def. 4 in the case $f = \chi_u$.

Definition 6 (Generalized solution) *Let $u_0 \in L^\infty(\mathbb{R})$, $0 \leq u_0 \leq 1$ a.e. A function $f \in L^1(Q\times\mathbb{R}_\xi)$ is said to be a generalized (kinetic) solution to (16)-(17) if*

$$0 \leq f \leq \chi_1 \text{ a.e.}, \; -\partial_\xi f_+ \text{ is a Young measure } Q \to \mathbb{R},$$

and if there exists non-negative measures $m_\pm$ on $[0,T]\times\mathbb{R}\times\mathbb{R}$ such that

- *m_+ is supported in $[0,T]\times\mathbb{R}\times]-\infty,1]$, m_- is supported in $[0,T]\times\mathbb{R}\times[0,+\infty[$,*
- *for all $\psi \in C_c^\infty([0,T[\times\mathbb{R}\times\mathbb{R})$,*

$$\int_Q\int_{\mathbb{R}} f_\pm(\partial_t\psi + k(x)a(\xi)\partial_x\psi)d\xi dtdx$$
$$+ \int_{\mathbb{R}}\int_{\mathbb{R}} f_{0,\pm}\psi(0,x,\xi)d\xi dx - (k_L-k_R)^\pm\int_0^T\int_{\mathbb{R}} a(\xi)\psi(t,0,\xi)d\xi dt$$
$$= \int_Q\int_{\mathbb{R}}\partial_\xi\psi dm_\pm(t,x,\xi) \quad (19)$$

where $f_{0,\pm}(x,\xi) = \mathrm{sgn}_\pm(u_0(x)-\xi)$.

Theorem 7 (Reduction, Uniqueness) *Let $u_0 \in L^\infty(\mathbb{R})$, $0 \leq u_0 \leq 1$ a.e. Problem (16)-(17) admits at most one solution. Besides, any generalized solution is actually a solution: if $f \in L^1(Q\times\mathbb{R}_\xi)$ is a generalized solution to (16)-(17), then there exists $u \in L^\infty(Q)$ such that $f = \chi_u$.*

To prepare the proof of Theorem 7, we first have to analyze the formulation (19) and the behavior of f at $t=0$ and $x=0$.

3.1 Weak traces

Introduce the cut-off function

$$\omega_\varepsilon(s) = \int_0^{|s|} \rho_\varepsilon(r)dr, \quad \rho_\varepsilon(s) = \varepsilon^{-1}\rho(\varepsilon^{-1}s), \quad s \in \mathbb{R}, \tag{20}$$

where $\rho \in C_c^\infty(\mathbb{R})$ is a non-negative function with total mass 1 compactly supported in $]0,1[$. We have the following proposition.

Proposition 8 (Weak traces) *Let $f \in L^\infty(Q \times \mathbb{R}_\xi)$ be a generalized solution to (16)-(17). There exists $f_\pm^{\tau_0} \in L^2(\mathbb{R}\times\mathbb{R})$, $F_\pm \in L^2(]0,T[\times\mathbb{R})$ and a sequence $(\eta_n) \downarrow 0$ such that, for all $\varphi \in L_c^2(\mathbb{R}\times\mathbb{R})$, for all $\theta \in L_c^2(]0,T[\times\mathbb{R})$ (the subscript c denotes compact support),*

$$\int_Q \int_{\mathbb{R}} f_\pm(t,x,\xi)\omega'_{\eta_n}(t)\varphi(x,\xi)d\xi dt dx \to \int_{\mathbb{R}}\int_{\mathbb{R}} f_\pm^{\tau_0}(x,\xi)\varphi(x,\xi)d\xi dx, \tag{21}$$

$$\int_Q \int_{\mathbb{R}} f_\pm(t,x,\xi)k(x)a(\xi)\omega'_{\eta_n}(x)\theta(t,\xi)d\xi dt dx \to \int_0^T \int_{\mathbb{R}} F_\pm(t,\xi)\theta(t,\xi)d\xi dt \tag{22}$$

as $n \to +\infty$. Besides, there exists non-negative measures $m_\pm^{\tau_0}$, $\bar{m}_\pm$ on $\mathbb{R}^2$ and $[0,T]\times\mathbb{R}$ respectively such that:

- *$m_+^{\tau_0}$ (resp. $\bar{m}_+$) is supported in $\mathbb{R}\times]-\infty,1]$ (resp. $[0,T]\times]-\infty,1]$), $m_-^{\tau_0}$ (resp. $\bar{m}_-$) is supported in $\mathbb{R}\times[0,+\infty[$ (resp. $[0,T]\times[0,+\infty[$),*
- *for all $\varphi \in C_c^\infty(\mathbb{R}^2)$, $\theta \in C_c^\infty([0,T[\times\mathbb{R})$,*

$$\int_{\mathbb{R}^2} f_\pm^{\tau_0}\varphi dx d\xi = \int_{\mathbb{R}^2} f_{0,\pm}\varphi dx d\xi - \int_{\mathbb{R}^2} \partial_\xi\varphi dm_\pm^{\tau_0}(x,\xi), \tag{23}$$

$$\begin{aligned}\int_0^T \int_{\mathbb{R}} F_\pm \theta d\xi dt = &-(k_L - k_R)^\pm \int_0^T \int_{\mathbb{R}} a(\xi)\theta d\xi dt \\ &- \int_0^T \int_{\mathbb{R}} \partial_\xi\theta d\bar{m}_\pm(t,\xi).\end{aligned} \tag{24}$$

Proof: The first part of the proposition does not use the fact that f is solution. Indeed, since $|f_\pm| \le 2$, we have

$$\left|\int_0^T f_\pm(t,x,\xi)\omega'_\eta(t)dt\right| \le 2\int_0^T |\omega'_\eta(t)|dt = 2\int_0^T \rho_\eta(t)dt \le 2,$$

for all $(x,\xi) \in \mathbb{R}^2$. This gives in particular a bound in $L^2(K)$, K compact of $\mathbb{R}^2$ on $\int_0^T f_\pm(t,\cdot)\omega'_\eta(t)dt$, hence existence of a subsequence that converges weakly in $L^2(K)$. Writing $\mathbb{R}^2$ as an increasing countable union of compact sets and using a diagonal process, we obtain (21). The proof of (22) is similar. To obtain (23), apply the formulation (19) to $\psi(t,x,\xi) = \varphi(x,\xi)(1-\omega_{\eta_n}(t))$. We obtain (23) by using (21) and setting

$$\int_{\mathbb{R}^2} \varphi dm_\pm^{\tau_0}(x,\xi) = \lim_{n\to+\infty} \int_Q \int_{\mathbb{R}} \varphi(x,\xi)(1-\omega_{\eta_n}(t))dm_\pm(t,x,\xi)$$

for all non-negative $\varphi \in C_c(\mathbb{R}^2)$: the limit is well defined since the argument is monotone in n and it defines a non-negative functional on $C_c(\mathbb{R}^2)$ which is represented by a non-negative Radon measure. Similarly, applying the formulation (19) to $\psi(t,x,\xi) = \theta(t,\xi)(1-\omega_{\eta_n}(x))$, we obtain (24) with

$$\int_0^T \int_{\mathbb{R}} \theta d\bar{m}_\pm(t,\xi) = \lim_{n\to+\infty} \int_Q \int_{\mathbb{R}} \theta(t,\xi)(1-\omega_{\eta_n}(x)) dm_\pm(t,x,\xi)$$

for all non-negative $\theta \in C_c([0,T]\times\mathbb{R})$. ■

Remark: Since $0 \le f \le \chi_1$, (21) shows that $f_+^{\tau_0}$, *resp.* $f_-^{\tau_0}$, is supported in $\mathbb{R}\times]-\infty,1]$, *resp.* $\mathbb{R}\times[0,+\infty[$. Similarly, F_+, *resp.* F_-, is supported in $[0,T]\times]-\infty,1]$, *resp.* $[0,T]\times[0,+\infty[$. We use this remark to show the following

Corollary 9 *For all $\varphi_- \in L^\infty(\mathbb{R}^2)$ supported in $[-R,R]\times[-R,+\infty[$ ($R>0$) such that $\partial_\xi\varphi_- \le 0$ (in the sense of distributions), we have*

$$\lim_{n\to+\infty} \int_Q \int_{\mathbb{R}} f_+ \omega'_{\eta_n}(t)\varphi_-(x,\xi) d\xi dt dx \ge \int_{\mathbb{R}^2} f_{0,+}\varphi_- dx d\xi. \tag{25}$$

For all $\theta_- \in L^\infty(]0,T[\times\mathbb{R})$ supported in $[0,T]\times[-R,+\infty[$ ($R>0$) such that $\partial_\xi\theta_- \le 0$ (in the sense of distributions), we have

$$\lim_{n\to+\infty} \int_Q \int_{\mathbb{R}} f_+ k(x) a(\xi) \omega'_{\eta_n}(x)\theta_-(t,\xi) d\xi dt dx \ge -(k_L-k_R)^+ \int_0^T \int_{\mathbb{R}} a(\xi)\theta_- d\xi dt. \tag{26}$$

Proof: Note first that each term in (25) is well defined by the remark above and that, by (21),

$$\lim_{n\to+\infty} \int_Q \int_{\mathbb{R}} f_+(t,x,\xi)\omega'_{\eta_n}(t)\varphi_-(x,\xi) d\xi dt dx = \int_{\mathbb{R}} \int_{\mathbb{R}} f_+^{\tau_0}\varphi_- d\xi dx.$$

By regularization (parameter ε) and truncation (parameter M), we have

$$\int_{\mathbb{R}^2} (f_+^{\tau_0} - f_{0,+})\varphi_- dx d\xi = \int_{\mathbb{R}^2} (f_+^{\tau_0} - f_{0,+})\varphi_-^{\varepsilon,M} dx d\xi + \eta(\varepsilon,M),$$

where $\lim\limits_{\varepsilon\to0, M\to+\infty} \eta(\varepsilon,M) = 0$. More precisely, we set

$$\varphi_-^{\varepsilon,M} = (\varphi_- * \psi_\varepsilon)\times\chi_M,$$

where ψ_ε is a (smooth, compactly supported) approximation of the unit on $\mathbb{R}^2$ and χ_M is a smooth, non-increasing function such that $\chi_M \equiv 1$ on $]-\infty,M]$, $\chi_M \equiv 0$ on $[M+1,+\infty[$. Apply (23) to $\varphi_-^{\varepsilon,M}$ to obtain

$$\int_{\mathbb{R}^2} (f_+^{\tau_0} - f_{0,+})\varphi_- dx d\xi = -\int_{\mathbb{R}^2} \partial_\xi\varphi_-^{\varepsilon,M} dm_+^{\tau_0}(x,\xi) + \eta(\varepsilon,M).$$

For $M > R+1$ and $\varepsilon < 1$, we have $\varphi_-^{\varepsilon,M} = \varphi_- * \psi_\varepsilon$, hence $\partial_\xi\varphi_-^{\varepsilon,M} \le 0$. It follows that

$$\int_{\mathbb{R}^2} (f_+^{\tau_0} - f_{0,+})\varphi_- dx d\xi \ge \eta(\varepsilon,M),$$

for $M > R+1$, $\varepsilon < 1$. At the limit $M\to+\infty$, $\varepsilon\to0$, we obtain (25). The proof of (26) is similar. ■

3.2 Proof of Theorem 7

Our aim is to show the following

Proposition 10 *Let $u_0, v_0 \in L^\infty(\mathbb{R})$, $0 \le u_0, v_0 \le 1$ a.e. and let f, resp g, be a generalized solution to (16)-(17) with datum u_0, resp. v_0. Let $M = \sup_{x\in\mathbb{R},\xi\in[0,1]} |k(x)a(\xi)|$. Then we have, for $R > 0$,*

$$\frac{1}{T}\int_0^T \int_{\{|x|<R\}} \int_{\mathbb{R}} -f_+ g_- d\xi dx dt \le \int_{\{|x|<R+MT\}} (u_0 - v_0)^+ dx. \tag{27}$$

Remark: In case $f = \chi_u$, $g = \chi_v$, we have $\int_{\mathbb{R}} -f_+ g_- d\xi = (u - v)^+$, hence (27) gives uniqueness of the solution to (16)-(17) (more precisely, it gives the L^1-contraction with averaging in time and the comparison result $u_0 \le v_0$ a.e. $\Rightarrow u \le v$ a.e.).
Remark: To obtain the second part of Theorem 7, we apply (27) with $g = f$ to obtain

$$\int_0^T \int_{\{|x|<R\}} \int_{\mathbb{R}} -f_+ f_- d\xi dx dt \le 0. \tag{28}$$

Since $0 \le f \le \chi_1$, we have $f_+ \ge 0$ a.e. and $f_- \le 0$ a.e. We deduce from (28) that $f_+ f_- = 0$ a.e. Let $\nu_{t,x}$ denote the Young measure $-\partial_\xi f_+$: we have $\partial_\xi f_- = \partial_\xi f - \delta_0 = \partial_\xi f_+$ and, by examination of the values at $\xi = \pm\infty$ of $f_\pm$, for a.e. $(t, x) \in Q$,

$$f_+(t, x, \xi) = \nu_{t,x}(\xi, +\infty), \quad f_-(t, x, \xi) = -\nu_{t,x}(-\infty, \xi).$$

But then, the relation $f_+ f_- = 0$ implies that $\nu_{t,x}$ is a Dirac mass at, say, $u(t, x)$. By measurability of ν, u is measurable and $f = \chi_u$.

Proof of Proposition 10: Since f_+ and g_- satisfy

$$\begin{aligned}\int_Q \int_{\mathbb{R}} & f_+(\partial_t\psi + k(x)a(\xi)\partial_x\psi) d\xi dt dx \\ & + \int_{\mathbb{R}} \int_{\mathbb{R}} f_{0,+}\psi(0, x, \xi) d\xi dx - (k_L - k_R)^+ \int_0^T \int_{\mathbb{R}} a(\xi)\psi(t, 0, \xi) d\xi dt \\ & = \int_Q \int_{\mathbb{R}} \partial_\xi\psi dm_+(t, x, \xi)\end{aligned} \tag{29}$$

and

$$\begin{aligned}\int_Q \int_{\mathbb{R}} & g_-(\partial_t\psi + k(x)a(\xi)\partial_x\psi) d\xi dt dx \\ & + \int_{\mathbb{R}} \int_{\mathbb{R}} g_{0,-}\psi(0, x, \xi) d\xi dx - (k_L - k_R)^- \int_0^T \int_{\mathbb{R}} a(\xi)\psi(t, 0, \xi) d\xi dt \\ & = \int_Q \int_{\mathbb{R}} \partial_\xi\psi dp_-(t, x, \xi)\end{aligned} \tag{30}$$

for all $\psi \in C_c^\infty([0, T[\times\mathbb{R}\times\mathbb{R})$ (here $g_{0,-} = \mathrm{sgn}_-(v_0 - \xi)$ and p_- is a non-negative measure on $[0, T] \times \mathbb{R} \times \mathbb{R}$ supported in $[0, T] \times \mathbb{R} \times [0, +\infty[$), it is possible to obtain an estimate for $-f_+g_-$ by setting $\psi = -g_-\varphi$ in (29) and $\psi = f_+\varphi$ in (30) (φ being a given test function) and adding the result. This requires first, however, a step of regularization.

Step 1. Regularization. Let $\rho_{\alpha,\varepsilon,\delta}$ denote the approximation of the unit on $\mathbb{R}^3$ given by

$$\rho_{\alpha,\varepsilon,\delta}(t,x,\xi) = \rho_\alpha(t)\rho_\varepsilon(x)\rho_\delta(\xi), \quad (t,x,\xi) \in \mathbb{R}^3,$$

where ρ_ε is defined in (20). Let $\psi \in C_c^\infty([0,T[\times\mathbb{R}\times\mathbb{R})$ be compactly supported in $]0,T[\times\mathbb{R}\setminus\{0\}\times\mathbb{R}$. Use $\psi * \rho_{\alpha,\varepsilon,\delta}$ as a test function in (29) and Fubini's theorem to obtain

$$\begin{aligned}
&\int_Q\int_{\mathbb{R}} f_+^{\alpha,\varepsilon,\delta}(\partial_t\psi + k(x)a(\xi)\partial_x\psi)d\xi dtdx \\
&+\int_{\mathbb{R}}\int_{\mathbb{R}} f_{0,+}\psi * \rho_{\alpha,\varepsilon,\delta}(0,x,\xi)d\xi dx - (k_L-k_R)^+\int_0^T\int_{\mathbb{R}} a(\xi)\psi * \rho_{\alpha,\varepsilon,\delta}(t,0,\xi)d\xi dt \\
&\qquad = \int_Q\int_{\mathbb{R}} \partial_\xi\psi dm_+^{\alpha,\varepsilon,\delta}(t,x,\xi) + R_{\alpha,\varepsilon,\delta}(\psi),
\end{aligned}$$

where $f_+^{\alpha,\varepsilon,\delta} := f_+ * \check\rho_{\alpha,\varepsilon,\delta}$, $m_+^{\alpha,\varepsilon,\delta} := m_+ * \check\rho_{\alpha,\varepsilon,\delta}$ and

$$R_{\alpha,\varepsilon,\delta}(\psi) = \int_Q\int_{\mathbb{R}} f_+[k(x)a(\xi)(\partial_x\psi) * \rho_{\alpha,\varepsilon,\delta} - (k(x)a(\xi)\partial_x\psi) * \rho_{\alpha,\varepsilon,\delta}]d\xi dtdx.$$

Here we have denoted $\check\rho(t,x,\xi) = \rho(-t,-x,-\xi)$. Also observe that, implicitly, we have extended f_+ by 0 outside $[0,T]$ since, *e.g.*

$$\begin{aligned}
\int_0^T f_+(t)\psi * \rho_\alpha(t)dt &= \int_0^T\int_{\mathbb{R}} f_+(t)\psi(s)\rho_\alpha(t-s)dsdt \\
&= \int_{\mathbb{R}}\psi(s)\int_0^T f_+(t)\check\rho_\alpha(s-t)dtds.
\end{aligned}$$

Since ψ is supported in $]0,T[\times\mathbb{R}\setminus\{0\}\times\mathbb{R}$, we have, for α,ε small enough,

$$\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}} f_{0,+}\psi * \rho_{\alpha,\varepsilon,\delta}(0,x,\xi)d\xi dx &=0, \\
\int_0^T\int_{\mathbb{R}} a(\xi)\psi * \rho_{\alpha,\varepsilon,\delta}(t,0,\xi)d\xi dt &=0,
\end{aligned}$$

and

$$R_{\alpha,\varepsilon,\delta}(\psi) = \int_Q\int_{\mathbb{R}} f_+k(x)[a(\xi)(\partial_x\psi) * \rho_{\alpha,\varepsilon,\delta} - (a(\xi)\partial_x\psi) * \rho_{\alpha,\varepsilon,\delta}]d\xi dtdx.$$

We deduce

$$\begin{aligned}
&\int_Q\int_{\mathbb{R}} f_+^{\alpha,\varepsilon,\delta}(\partial_t\psi + k(x)a(\xi)\partial_x\psi)d\xi dtdx \\
&\qquad = \int_Q\int_{\mathbb{R}} \partial_\xi\psi dm_+^{\alpha,\varepsilon,\delta}(t,x,\xi) + R_{\alpha,\varepsilon,\delta}(\psi). \quad (31)
\end{aligned}$$

A similar work on g_- gives

$$\int_Q \int_{\mathbb{R}} g_-^{\beta,\nu,\sigma}(\partial_t \psi + k(x)a(\xi)\partial_x \psi) d\xi dt dx$$
$$= \int_Q \int_{\mathbb{R}} \partial_\xi \psi dp_-^{\beta,\nu,\sigma}(t,x,\xi) + Q_{\beta,\nu,\sigma}(\psi), \quad (32)$$

where

$$Q_{\beta,\nu,\sigma}(\psi) = \int_Q \int_{\mathbb{R}} g_- k(x)[a(\xi)(\partial_x \psi) * \rho_{\beta,\nu,\sigma} - (a(\xi)\partial_x \psi) * \rho_{\beta,\nu,\sigma}] d\xi dt dx.$$

Step 2. Equation for $-f_+^{\alpha,\varepsilon,\delta} g_-^{\beta,\nu,\sigma}$. Let $\varphi \in C_c^\infty([0,T[\times\mathbb{R})$ be non-negative and compactly supported in $]0,T[\times\mathbb{R}\setminus\{0\}$. Notice that φ does not depend on ξ. Set $\psi = -\varphi g_-^{\beta,\nu,\sigma}$ in (31), $\psi = -\varphi f_+^{\alpha,\varepsilon,\delta}$ in (32). Since

$$f\partial_t(\varphi g) + g\partial_t(\varphi f) = fg\partial_t \varphi + \partial_t(\varphi f g),$$

we obtain by addition of the resulting equations

$$\int_Q \int_{\mathbb{R}} -f_+^{\alpha,\varepsilon,\delta} g_-^{\beta,\nu,\sigma}(\partial_t \varphi + k(x)a(\xi)\partial_x \varphi) d\xi dt dx$$
$$= -\int_Q \varphi \int_{\mathbb{R}} \partial_\xi f_+^{\alpha,\varepsilon,\delta} dp_-^{\beta,\nu,\sigma}(t,x,\xi) + \partial_\xi g_-^{\beta,\nu,\sigma} dm_+^{\alpha,\varepsilon,\delta}(t,x,\xi)$$
$$+ R_{\alpha,\varepsilon,\delta}(-\varphi g_-^{\beta,\nu,\sigma}) + Q_{\beta,\nu,\sigma}(-\varphi f_+^{\alpha,\varepsilon,\delta}).$$

Notice that the term

$$-\int_Q \varphi \int_{\mathbb{R}} \partial_\xi f_+^{\alpha,\varepsilon,\delta} dp_-^{\beta,\nu,\sigma}(t,x,\xi) + \partial_\xi g_-^{\beta,\nu,\sigma} dm_+^{\alpha,\varepsilon,\delta}(t,x,\xi)$$

is well defined since the intersection of the supports of the functions $f_+^{\alpha,\varepsilon,\delta}$ and $p_-^{\beta,\nu,\sigma}$ (*resp.* $f_-^{\beta,\nu,\sigma}$ and $m_+^{\alpha,\varepsilon,\delta}$) is compact. Actually, this term is non-negative since $p_-^{\beta,\nu,\sigma}, m_+^{\alpha,\varepsilon,\delta} \geq 0$ and $\partial_\xi f_+^{\alpha,\varepsilon,\delta}, \partial_\xi g_-^{\beta,\nu,\sigma} \leq 0$. We thus have

$$\int_Q \int_{\mathbb{R}} -f_+^{\alpha,\varepsilon,\delta} g_-^{\beta,\nu,\sigma}(\partial_t \varphi + k(x)a(\xi)\partial_x \varphi) d\xi dt dx$$
$$\geq R_{\alpha,\varepsilon,\delta}(-\varphi g_-^{\beta,\nu,\sigma}) + Q_{\beta,\nu,\sigma}(-\varphi f_+^{\alpha,\varepsilon,\delta}). \quad (33)$$

It is easily checked that

$$R_{\alpha,\varepsilon,\delta}(-\varphi j_-^{\beta,\nu,\sigma}) = \mathcal{O}(\nu^{-1}\delta), \quad Q_{\beta,\nu,\sigma}(-\varphi h_+^{\alpha,\varepsilon,\delta}) = \mathcal{O}(\varepsilon^{-1}\sigma),$$

hence

$$\lim_{\delta,\sigma\to 0} R_{\alpha,\varepsilon,\delta}(-\varphi g_-^{\beta,\nu,\sigma}) + Q_{\beta,\nu,\sigma}(-\varphi f_+^{\alpha,\varepsilon,\delta}) = 0.$$

At the limit $\delta, \sigma \to 0$ in (33), we conclude that

$$\int_Q \int_{\mathbb{R}} -f_+^{\alpha,\varepsilon} g_-^{\beta,\nu}(\partial_t \varphi + k(x)a(\xi)\partial_x \varphi) d\xi dt dx \geq 0. \quad (34)$$

Step 3. Traces. Suppose that $k_L < k_R$. We then pass to the limit $\varepsilon, \alpha \to 0$ in (34) to obtain

$$\int_Q \int_{\mathbb{R}} -f_+ g_-^{\beta,\nu} (\partial_t \varphi + k(x) a(\xi) \partial_x \varphi) d\xi dt dx \geq 0. \tag{35}$$

Note that in the opposite case $k_L > k_R$, and with our method of proof, we would *first* pass to the limit on β, ν. Let us now remove the hypothesis that φ vanishes at $t = 0$: suppose that $\psi \in C_c^\infty([0, T[\times\mathbb{R})$ is non-negative and supported in $[0, T[\times\mathbb{R} \setminus \{0\}$ and apply (35) to $\varphi(t, x) = \psi(t, x)\omega_{\eta_n}(t)$. We have

$$\int_Q \int_{\mathbb{R}} -f_+ g_-^{\beta,\nu} \omega_{\eta_n}(t) (\partial_t \psi + k(x) a(\xi) \partial_x \psi) d\xi dt dx \\ + \int_Q \int_{\mathbb{R}} -f_+ g_-^{\beta,\nu} \psi(t, x) \omega'_{\eta_n}(t) d\xi dt dx \geq 0. \tag{36}$$

By (25) applied with $\varphi_-(x, \xi) = g_-^{\beta,\nu}(0, x, \xi)\psi(0, x)$, we obtain

$$\lim_{n\to+\infty} \int_Q \int_{\mathbb{R}} f_+ g_-^{\beta,\nu}(0, x, \xi)\psi(0, x)\omega'_{\eta_n}(t) d\xi dt dx \\ \geq \int_{\mathbb{R}} \int_{\mathbb{R}} f_{0,+} g_-^{\beta,\nu}(0, x, \xi)\psi(0, x) d\xi dx.$$

Now $f_+(t, x, \xi) g_-^{\beta,\nu}(t, x, \xi)\psi(t, x)$ has a compact support, say in $[0, T]\times[-R, R]\times[-R, R]$, thus $\varphi_-(t, x, \xi) = g_-^{\beta,\nu}(t, x, \xi)\psi(t, x)$ is uniformly continuous on this compact support. Therefore for $\mu > 0$, there exists $\gamma > 0$ such that $|\varphi_-(t, x, \xi) - \varphi_-(0, x, \xi)| \leq \frac{\mu}{8R^2}$ for any $0 \leq t < \gamma$ and any $x, \xi \in [-R, R]$, and then for large n, we have $\eta_n < \gamma$ and

$$\left| \int_Q \int_{\mathbb{R}} f_+(t, x, \xi) \left(g_-^{\beta,\nu}(t, x, \xi)\psi(t, x) - g_-^{\beta,\nu}(0, x, \xi)\psi(0, x) \right) \omega'_{\eta_n}(t) d\xi dt dx \right| \\ \leq \int_Q \int_{\mathbb{R}} |f_+(t, x, \xi)| \rho_{\eta_n}(t) \frac{\mu}{8R^2} \mathbb{1}_{(x,\xi)\in[-R,R]^2} d\xi dt dx \\ \leq \mu \int \rho_{\eta_n}(t)\, dt = \mu. \tag{37}$$

Thus we obtain, at the limit $n \to +\infty$ in (36),

$$\int_Q \int_{\mathbb{R}} -f_+ g_-^{\beta,\nu} (\partial_t \psi + k(x) a(\xi) \partial_x \psi) d\xi dt dx \\ + \int_{\mathbb{R}} \int_{\mathbb{R}} -f_{0,+} g_-^{\beta,\nu}(0, x, \xi)\psi(0, x) d\xi dx \geq 0.$$

The next step is then to remove the hypothesis that ψ vanishes at $x = 0$ by setting $\psi(t, x) = \theta(t, x)\omega_{\eta_n}(x)$ where $\theta \in C_c^\infty([0, T[\times\mathbb{R})$ is a non-negative test-function. We have

$$\int_Q\int_{\mathbb{R}} -f_+ g_-^{\beta,\nu}\omega_{\eta_n}(x)(\partial_t\theta + k(x)a(\xi)\partial_x\theta)d\xi dtdx + \int_Q\int_{\mathbb{R}} -f_+ g_-^{\beta,\nu}\theta(t,x)k(x)a(\xi)\omega'_{\eta_n}(x)d\xi dtdx + \int_{\mathbb{R}}\int_{\mathbb{R}} -f_{0,+}g_-^{\beta,\nu}(0,x,\xi)\theta(0,x)\omega_{\eta_n}(x)d\xi dx \geq 0.$$

By (26) with $\theta_-(t,\xi) = g_-^{\beta,\nu}(t,0,\xi)\theta(t,0)$,

$$\lim_{n\to+\infty}\int_Q\int_{\mathbb{R}} f_+ k(x)a(\xi)\omega'_{\eta_n}(x)g_-^{\beta,\nu}(t,0,\xi)\theta(t,0)d\xi dtdx \geq -(k_L-k_R)^+\int_0^T\int_{\mathbb{R}} a(\xi)g_-^{\beta,\nu}(t,0,\xi)\theta(t,0)d\xi dt,$$

and by an argument similar to (37), the limit as $[n\to+\infty]$ of the term

$$\int_Q\int_{\mathbb{R}} f_+ k(x)a(\xi)\omega'_{\eta_n}(x)\left(g_-^{\beta,\nu}(t,x,\xi)\theta(t,x) - g_-^{\beta,\nu}(t,0,\xi)\theta(t,0)\right)d\xi dtdx$$

is zero. We have therefore

$$\int_Q\int_{\mathbb{R}} -f_+ g_-^{\beta,\nu}(\partial_t\theta + k(x)a(\xi)\partial_x\theta)d\xi dtdx + (k_L-k_R)^+\int_0^T\int_{\mathbb{R}} a(\xi)g_-^{\beta,\nu}(t,0,\xi)\theta(t,0)d\xi dt + \int_{\mathbb{R}}\int_{\mathbb{R}} -f_{0,+}g_-^{\beta,\nu}(0,x,\xi)\theta(0,x)d\xi dx \geq 0.$$

Since $(k_L-k_R)^+ = 0$, we have actually

$$\int_Q\int_{\mathbb{R}} -f_+ g_-^{\beta,\nu}(\partial_t\theta + k(x)a(\xi)\partial_x\theta)d\xi dtdx + \int_{\mathbb{R}}\int_{\mathbb{R}} -f_{0,+}g_-^{\beta,\nu}(0,x,\xi)\theta(0,x)d\xi dx \geq 0.$$

Take $\beta = \eta_n$ where (η_n) is given in Prop. 8. At the limit $\nu\to 0$ first, then $n\to+\infty$, we obtain

$$\int_Q\int_{\mathbb{R}} -f_+ g_-(\partial_t\theta + k(x)a(\xi)\partial_x\theta)d\xi dtdx + \limsup_{n\to+\infty}\int_{\mathbb{R}}\int_{\mathbb{R}} -f_{0,+}g_-^{\eta_n}(0,x,\xi)\theta(0,x)d\xi dx \geq 0. \quad (38)$$

Observe that

$$\begin{aligned} g_-^{\eta_n}(0,x,\xi) &= \int_0^T g_-(t,x,\xi)\rho_{\eta_n}(t)dt \\ &= \int_0^T g_-(t,x,\xi)\omega'_{\eta_n}(t)dt. \end{aligned}$$

By (25) (transposed to g_- tested against a function φ_+), we have

$$\lim_{n\to+\infty}\int_{\mathbb{R}}\int_{\mathbb{R}} -f_{0,+}g_-^{\eta_n}(0,x,\xi)\theta(0,x)d\xi dx \leq \int_{\mathbb{R}}\int_{\mathbb{R}} -f_{0,+}g_{0,-}\theta(0,x)d\xi dx.$$

Since

$$\int_{\mathbb{R}} -f_{0,+}g_{0,-}d\xi = \int_{\mathbb{R}} -\mathrm{sgn}_+(u_0-\xi)\mathrm{sgn}_-(v_0-\xi)d\xi = (u_0-v_0)^+,$$

we obtain by (38),

$$\int_Q\int_{\mathbb{R}} -f_+g_-(\partial_t\theta + k(x)a(\xi)\partial_x\theta)d\xi dtdx + \int_{\mathbb{R}}(u_0-v_0)^+\theta(0,x)dx \geq 0. \quad (39)$$

It is then classical to conclude to (27): let $M>0, R>MT$, let $\eta>0$ and let r be a non-negative, non-increasing function such that $r\equiv 1$ on $[0,R]$, $r\equiv 0$ on $[R+\eta,+\infty[$. Set $\theta(t,x)=\frac{T-t}{T}r(|x|+Mt)$ in (39) to obtain

$$\frac{1}{T}\int_Q\int_{\mathbb{R}} -f_+g_-r(|x|+Mt)d\xi dtdx \leq \int_{\{|x|\leq R+\eta\}}(u_0-v_0)^+dx + \mathrm{J},$$

where the remainder term is

$$\mathrm{J} = \int_Q\int_{\mathbb{R}} -f_+g_-\frac{T-t}{T}r'(|x|+Mt)(M+k(x)a(\xi)\mathrm{sgn}(x))d\xi dxdt.$$

By definition of M, $\mathrm{J}\leq 0$ and since $r(|x|+Mt)=1$ for $|x|\leq R-MT$, $0\leq t\leq T$, we obtain

$$\frac{1}{T}\int_0^T\int_{|x|<R-MT}\int_{\mathbb{R}} -f_+g_-d\xi dxdt \leq \int_{\{|x|\leq R+\eta\}}(u_0-v_0)^+dx.$$

Replacing R by $R+MT$, and letting $\eta\to 0$ gives (27). ■

4 Convergence of the BGK approximation

Theorem 11 *Let $u_0\in L^1\cap L^\infty(\mathbb{R})$, $0\leq u_0\leq 1$ a.e. When $\varepsilon\to 0$, the solution f^ε to the (1) with initial datum $f_0=\chi_{u_0}$ converges in $L^p(Q\times\mathbb{R}_\xi)$, $1\leq p<+\infty$ to χ_u, where u is the unique solution to (16)-(17).*

Proof: For $f\in L^1(\mathbb{R}_\xi)$, set

$$m_f(\xi) = \int_{-\infty}^{\xi}(\chi_u - f)(\zeta)d\zeta, \quad u = \int_{\mathbb{R}} f(\xi)d\xi.$$

It is easy to check that $m_f\geq 0$ if $0\leq \mathrm{sgn}(\xi)f(\xi)\leq 1$ for a.e. ξ (*cf.* (29) in [Bre83]). In our context, we have $0\leq f^\varepsilon\leq \chi_1$, hence $m^\varepsilon := \frac{1}{\varepsilon}m_{f^\varepsilon}\geq 0$. Viewed as a measure, m^ε is supported in $[0,T]\times\mathbb{R}_x\times[0,1]$. Integration with respect to ξ in (1) gives

$$m^\varepsilon(\xi) = \partial_t\left(\int_0^\xi f^\varepsilon(\zeta)d\zeta\right) + \partial_x\left(k(x)\int_0^\xi a(\zeta)f^\varepsilon(\zeta)d\zeta\right)$$

in $\mathcal{D}'(]0,T[\times\mathbb{R}_x)$. Summing over $(t,x)\in[0,T]\times[x_1,x_2]$, $\xi\in]0,1[$, we get the estimate

$$\begin{aligned} m^\varepsilon([0,T]\times[x_1,x_2]\times[0,1]) &= \int_{x_1}^{x_2}\int_0^1(1-\xi)(f^\varepsilon(T,x,\xi)-f^\varepsilon(0,x,\xi))d\xi dx \\ &\quad+\left[\int_0^T\int_0^1(1-\xi)k(x)a(\xi)f^\varepsilon(t,x,\xi)d\xi dt\right]_{x_1}^{x_2}. \end{aligned} \tag{40}$$

Since $f^\varepsilon(t)\in L^1(\mathbb{R}_x\times\mathbb{R}_\xi)$, there exists sequences $(x_1^n)\downarrow-\infty$ and $(x_2^n)\uparrow+\infty$ such that the last term of the right hand-side in (40) tends to 0 when $n\to+\infty$. Since, besides, $f^\varepsilon\geq 0$ and

$$\int_{\mathbb{R}}\int_{\mathbb{R}} f^\varepsilon(T,x,\xi)d\xi dx\leq\int_{\mathbb{R}}\int_{\mathbb{R}}\chi_{u_0}d\xi dx=\|u_0\|_{L^1(\mathbb{R})},$$

we obtain the uniform estimate

$$m^\varepsilon([0,T]\times\mathbb{R}\times[0,1])\leq\|u_0\|_{L^1(\mathbb{R})}. \tag{41}$$

We also have

$$0\leq f^\varepsilon\leq\chi_1,\quad -\partial_\xi f_+^\varepsilon(t,x,\xi)=\nu_{t,x}^\varepsilon(\xi)+\mathcal{O}(\varepsilon) \tag{42}$$

where $\nu_{t,x}^\varepsilon(\xi):=\delta_{u^\varepsilon(t,x)}(\xi)$ and the identity is satisfied in $\mathcal{D}'(]0,T[\times\mathbb{R}_x\times\mathbb{R}_\xi)$. Indeed, by (1),

$$f^\varepsilon=\chi_{u^\varepsilon}+\varepsilon(\partial_t f^\varepsilon+\partial_x(k(x)a(\xi)f^\varepsilon))=\chi_{u^\varepsilon}+\mathcal{O}(\varepsilon),$$

hence

$$-\partial_\xi f_+^\varepsilon=-\partial_\xi f^\varepsilon+\delta_0(\xi)=-\partial_\xi\chi_{u^\varepsilon}+\delta_0(\xi)+\mathcal{O}(\varepsilon)=\delta_{u^\varepsilon}(\xi)+\mathcal{O}(\varepsilon).$$

Notice that, for a.e. (t,x), $\nu_{t,x}^\varepsilon$ is supported in the fixed compact subset $[0,1]$ of $\mathbb{R}_\xi$. We deduce from (41)-(42) that, up to a subsequence, there exists a non-negative measure m on $\mathbb{R}^3$ supported in $[0,T]\times\mathbb{R}_x\times[0,1]$, a function $f\in L^\infty(]0,T[;L^1(\mathbb{R}_x\times\mathbb{R}_\xi))$ such that $0\leq f\leq\chi_1$, $-\partial_\xi f_+(t,x,\xi)=\nu_{t,x}(\xi)$ where ν is a Young measure $Q\to\mathbb{R}_\xi$ and such that $m^\varepsilon\rightharpoonup m$ weakly in the sense of measures (*i.e.* $\langle m^\varepsilon-m,\varphi\rangle\to 0$ for every continuous compactly supported φ on $\mathbb{R}^3$) and $f^\varepsilon\rightharpoonup f$ in $L^\infty(Q\times\mathbb{R}_\xi)$ weak-star. Besides, since $f^\varepsilon\in C([0,T];L^1_{x,\xi})$ satisfies $f^\varepsilon(0)=f_0$ and the BGK equation

$$\partial_t f^\varepsilon+\partial_x(k(x)a(\xi)f^\varepsilon)=\partial_\xi m^\varepsilon,$$

it satisfies the weak formulation: for all $\psi\in C_c^\infty([0,T[\times\mathbb{R}\times\mathbb{R})$,

$$\begin{aligned} \int_Q\int_{\mathbb{R}} f^\varepsilon(\partial_t\psi+k(x)a(\xi)\partial_x\psi)d\xi dt dx+\int_{\mathbb{R}}\int_{\mathbb{R}} f_0\psi(0,x,\xi)d\xi dx \\ =\int_Q\int_{\mathbb{R}}\partial_\xi\psi dm^\varepsilon(t,x,\xi). \end{aligned}$$

In particular, we have

$$\begin{aligned}
&\int_Q \int_{\mathbb{R}} f^\varepsilon_\pm (\partial_t \psi + k(x)a(\xi)\partial_x\psi) d\xi dt dx + \int_{\mathbb{R}}\int_{\mathbb{R}} f_{0,\pm}\psi(0,x,\xi)d\xi dx \\
=& -\int_Q\int_{\mathbb{R}} \mathrm{sgn}_\mp(\xi)k(x)a(\xi)\partial_x\psi d\xi dt dx + \int_Q\int_{\mathbb{R}} \partial_\xi \psi dm^\varepsilon(t,x,\xi) \\
=&(k_R-k_L)\int_0^T \int_{\mathbb{R}} \mathrm{sgn}_\mp(\xi)a(\xi)\psi(t,0,\xi)d\xi dt + \int_Q\int_{\mathbb{R}} \partial_\xi\psi dm^\varepsilon(t,x,\xi) \\
=&(k_L-k_R)^\pm \int_0^T \int_{\mathbb{R}} a(\xi)\psi(t,0,\xi)d\xi dt + \int_Q\int_{\mathbb{R}} \partial_\xi \psi dm^\varepsilon_\pm(t,x,\xi), \qquad (43)
\end{aligned}$$

where

$$\begin{aligned}
\langle m^\varepsilon_\pm, \partial_\xi\psi\rangle :=& \langle m^\varepsilon, \partial_\xi \psi\rangle \\
&- \int_0^T \int_{\mathbb{R}} a(\xi)[(k_L-k_R)\mathrm{sgn}_\mp(\xi) + (k_L-k_R)^\pm]\psi(t,0,\xi)d\xi dt. \qquad (44)
\end{aligned}$$

More precisely, we set

$$m^\varepsilon_+ = m^\varepsilon + \int_\xi^{+\infty} a(\zeta)[(k_L-k_R)^+\mathrm{sgn}_+(\zeta) - (k_L-k_R)^-\mathrm{sgn}_-(\zeta)]d\zeta\delta(x=0),$$

and

$$m^\varepsilon_- = m^\varepsilon + \int_{-\infty}^\xi a(\zeta)[(k_L-k_R)^+\mathrm{sgn}_+(\zeta) - (k_L-k_R)^-\mathrm{sgn}_-(\zeta)]d\zeta\delta(x=0).$$

Notice that in both cases, and since $A(\xi) \geq 0$ for any ξ, we have added a non-negative quantity to m^ε. At the limit $\varepsilon \to 0$ we thus obtain $m^\varepsilon_\pm \rightharpoonup m_\pm$ where $m_\pm$ is a non-negative measure. Examination of the support of $m^\varepsilon_\pm$ shows that m_+, *resp.* m_- is supported in $[0,T]\times\mathbb{R}_x\times]-\infty,1]$, *resp.* $[0,T]\times\mathbb{R}_x\times[0,+\infty[$. At the limit $\varepsilon\to 0$, we thus obtain the kinetic formulation (19). We conclude that f is a generalized solution to (16)-(17). By Theorem 7, $f=\chi_u$ where $u\in L^\infty(Q)$ is solution to (16)-(17). By uniqueness, the whole sequence (f^ε) converges (in L^∞ weak-star) to χ_u. Actually the convergence is strong since

$$\begin{aligned}
\int_Q\int_{\mathbb{R}} |f^\varepsilon - \chi_u|^2 d\xi dt dx &= \int_Q\int_{\mathbb{R}} |f^\varepsilon|^2 - 2f^\varepsilon\chi_u + \chi_u d\xi dt dx \\
&\leq \int_Q\int_{\mathbb{R}} f^\varepsilon - 2f^\varepsilon\chi_u + \chi_u d\xi dt dx. \qquad (45)
\end{aligned}$$

We have used the fact that $0\leq f^\varepsilon\leq 1$. The right-hand side of (45) tends to 0 when $\varepsilon\to 0$ since 1, $\chi_u\in L^\infty$ can be taken as test functions. Hence $f^\varepsilon\to\chi_u$ in $L^2(Q\times\mathbb{R})$. The convergence in $L^p(Q\times\mathbb{R})$, $1\leq p<+\infty$ follows from the uniform bound on f^ε in $L^1\cap L^\infty(Q\times\mathbb{R})$. ■

Remark: it is possible to relax the assumption that the initial datum for (1) is at equilibrium and independent on ε in Theorem 11. Indeed, the conclusion of Theorem 11 remains valid under the hypothesis that the initial datum f_0^ε for (1) satisfies

$$0 \leq f_0^\varepsilon \leq \chi_1, \quad f_0^\varepsilon \rightharpoonup f_0, \quad u_0(x) := \int_{\mathbb{R}} f_0(x,\xi)d\xi, \tag{46}$$

where $f_0^\varepsilon \rightharpoonup f_0$ in (46) denotes weak convergence in $L^1(\mathbb{R}_x \times \mathbb{R}_\xi)$. Indeed, the proof of Theorem 11 remains unchanged under the following modification: passing to the limit in (43), we obtain that f is a generalized solution to (16) with an initial datum f_0 that is not necessary at equilibrium. However, we have (*cf.* (29) in [Bre83])

$$f_0 - \mathrm{sgn}_\mp(\xi) = \mathrm{sgn}_\pm(u_0 - \xi) - \partial_\xi m^0_\pm,$$

where m^0_+ (*resp.* m^0_-) is a non-negative measure supported in $[0,T]\times\mathbb{R}\times]-\infty,1]$ (*resp.* $[0,T]\times\mathbb{R}\times[0,+\infty[$). Consequently, up to a modification of the kinetic measure $m_\pm$, we obtain that f is indeed a generalized solution to (16)-(17). The rest of the proof is similar.

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